

Regularization dependence on the Schwinger–Dyson equation in Abelian gauge theory: 4D vs 3D cutoff regularization

Hiroaki Kohyama

Department of Physics, National Taiwan University, Taipei 10617, Taiwan.

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Abstract

We study the regularization dependence on the quenched Schwinger–Dyson equations in general gauge by applying the two types of regularizations, the four and three dimensional momentum cutoffs. The obtained results indicate that the solutions are not drastically affected by the choice of two different cutoff prescriptions. We then think that both the regularizations can nicely be adopted in the analyses for the Schwinger–Dyson equations.

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I. INTRODUCTION

The chiral symmetry breaking in the strongly coupling system is interesting phenomena in quantum chromodynamics. Although the coupling strength is small in realistic quantum electrodynamics (QED), the chiral symmetry can be broken when we consider strongly coupled QED. Then it is interesting to study such a system in Abelian gauge theories.

For the investigation on the above mentioned chiral symmetry breaking coming from dynamical mass generation, the analysis based on the Schwinger–Dyson equation (SDE) may be an appropriate approach [1]. The SDE is the set of equations for green functions whose solutions give the information on the field renormalization and dynamical mass generation (for reviews, see, [2–4]). The analysis of the SDE should conceptually lead the gauge and regularization independent solutions, since the equations stem from a renormalizable gauge theory. However, the solutions practically depend on the chosen gauge, because the equations are derived through applying some approximations in the intermediate steps. The extensive analyses on the gauge dependence are given in [5] where the effects of generalised vertices are studied in unquenched QED with four dimensional cutoff method. Concerning on the similar generalisations of the equations, a lot of works have been done with various approaches, see, e.g., [6, 7]. Also, it is known that the physical predictions as well depend on regularization procedures [8], since the quenched SDE can be regarded as the generalised version of the Nambu Jona Lasinio-type gap equations in which regularization has effects on the model predictions [9]. We then think it may be interesting to study the regularization dependence on the solutions of the SDE.

In this letter, we shall numerically solve the SDE in general gauge with two regularization procedures, the four dimensional (4D) and three dimensional (3D) cutoff regularizations, then make the comparison between these two methods. The importance of studying the equations with the 3D cutoff regularization lies on the fact that the solutions can smoothly be continued to the ones obtained in the finite temperature system, since the equations are usually investigated by using the 3D cutoff scheme at finite temperature [10].

This paper is organised as follows; Section II presents two types of equations. We show the numerical results for the field renormalization factor and the dynamically generated mass in Sec. III. The concluding remarks are given in Sec. IV.

II. SCHWINGER-DYSON EQUATION

The Schwinger–Dyson equation for fermion self-energy $\Sigma(P)$ is written by

$$\Sigma(P) = ie^2 \int \frac{d^4 Q}{(2\pi)^4} \gamma^\mu D_{\mu\nu}(P-Q) S(Q) \Gamma^\nu(P, Q), \quad (1)$$

where e is the coupling strength, $D_{\mu\nu}(P-Q)$ and $S(Q)$ are the gauge boson and fermion propagators, and $\Gamma^\nu(P, Q)$ is the vertex function on the gauge boson and the fermion. In this letter, we use the capital letters as P, Q for expressing the four dimensional momenta, namely, $P_\mu = (p_0, \mathbf{p})$ and $Q_\mu = (q_0, \mathbf{q})$. For $D_{\mu\nu}$ and Γ^ν , we employ the following tree-level forms

$$D_{\mu\nu}(K) = \frac{-g_{\mu\nu}}{K^2} + (1 - \xi) \frac{K_\mu K_\nu}{K^4}, \quad (2)$$

$$\Gamma^\nu(P, Q) = \gamma^\nu, \quad (3)$$

with the gauge parameter ξ and $K_\mu = P_\mu - Q_\mu$.

A. Equations with the four dimensional cutoff

In the SDE with four dimensional cutoff, we define the fermion propagator by

$$S(Q) = \frac{1}{A(Q)Q_\mu \gamma^\mu - B(Q)}, \quad (4)$$

where $A(Q)$ and $B(Q)$ indicate the field strength factor and the mass function. The insertion of these quantities leads

$$A(P)P^\mu \gamma_\mu - B(P) = P^\mu \gamma_\mu - m_0 + e^2 \int \frac{d^4 Q}{i(2\pi)^4} \gamma^\mu \left[\frac{-g_{\mu\nu}}{K^2} + (1 - \xi) \frac{K_\mu K_\nu}{K^4} \right] \left(\frac{1}{A(Q)Q_\rho \gamma^\rho - B(Q)} \right) \gamma^\nu, \quad (5)$$

with the fermion bare mass m_0 appearing in the Lagrangian. Taking the trace after multiplying $P^\rho \gamma_\rho$ and without the multiplication give the equations for $A(P)$ and $B(P)$ as

$$A(P) = 1 - \frac{e^2}{P^2} \int \frac{d^4 Q}{i(2\pi)^4} \left[\xi_+ \frac{P \cdot Q}{K^2} + 2\xi_- \frac{(P \cdot K)(Q \cdot K)}{K^4} \right] \Delta'(Q)A(Q), \quad (6)$$

$$B(P) = m_0 - e^2 \int \frac{d^4 Q}{i(2\pi)^4} \left[\xi_3 \frac{1}{K^2} \right] \Delta'(Q)B(Q), \quad (7)$$

where $\xi_\pm \equiv 1 \pm \xi$, $\xi_3 \equiv 3 + \xi$ and

$$\Delta'(Q) = \frac{-1}{A^2(Q)Q^2 - B^2(Q)}. \quad (8)$$

Performing the angular integration after the Wick rotation, we obtain the following equations,

$$A(P) = 1 + \frac{\alpha\xi}{4\pi} \int_{\delta_{4D}^2}^{\Lambda_{4D}^2} dQ^2 \left[\frac{Q^4}{P^4} \theta(P - Q) + \theta(Q - P) \right] \Delta(Q) A(Q), \quad (9)$$

$$B(P) = m_0 + \frac{\alpha\xi_3}{4\pi} \int_{\delta_{4D}^2}^{\Lambda_{4D}^2} dQ^2 \left[\frac{Q^2}{P^2} \theta(P - Q) + \theta(Q - P) \right] \Delta(Q) B(Q), \quad (10)$$

with

$$\Delta(Q) = \frac{1}{A^2(Q)Q^2 + B^2(Q)}. \quad (11)$$

where $\alpha = e^2/(4\pi)$ and we introduce the ultraviolet and infrared cutoffs, Λ_{4D} and δ_{4D} . These are the equations with the four dimensional cutoff scheme.

B. Equations with the three dimensional cutoff

In the three dimensional cutoff, we need to consider the following fermion propagator

$$S(p_0, p) = \frac{1}{C(p_0, p)\gamma_0 p^0 + A(p_0, p)\gamma_i p^i - B(p_0, p)} \quad (12)$$

with $p = |\mathbf{p}|$, since the time and space directions should be treated separately. After the Wick rotation and a bit of algebras one obtains the following forms

$$C(p_0, p) = 1 + \frac{\alpha}{2\pi^2} \int_{-\infty}^{\infty} dq_0 \int_{\delta_{3D}}^{\Lambda_{3D}} dq [\mathcal{I}_{CA} A(q_0, q) + \mathcal{I}_{CC} C(q_0, q)] \Delta(q_0, q), \quad (13)$$

$$A(p_0, p) = 1 + \frac{\alpha}{2\pi^2} \int_{-\infty}^{\infty} dq_0 \int_{\delta_{3D}}^{\Lambda_{3D}} dq [\mathcal{I}_{AA} A(q_0, q) + \mathcal{I}_{AC} C(q_0, q)] \Delta(q_0, q), \quad (14)$$

$$B(p_0, p) = m_0 + \frac{\alpha}{2\pi^2} \int_{-\infty}^{\infty} dq_0 \int_{\delta_{3D}}^{\Lambda_{3D}} dq [\mathcal{I}_B B(q_0, q)] \Delta(q_0, q), \quad (15)$$

with

$$\Delta(q_0, q) = \frac{1}{C^2(q_0, q)q_0^2 + A^2(q_0, q)q^2 + B^2(q_0, q)}. \quad (16)$$

and

$$\mathcal{I}_{CA} = \xi_- \frac{k_0}{p_0} I_1 + \xi_- \frac{k_0}{p_0} [k_0^2 - q^2 + p^2] I_2, \quad (17)$$

$$\mathcal{I}_{CC} = -\xi_+ \frac{q_0}{p_0} I_1 + 2\xi_- \frac{q_0}{p_0} k_0^2 I_2, \quad (18)$$

$$\mathcal{I}_{AA} = -\frac{2q^2}{p^2} - \frac{1}{2p^2} [\xi_3 k_0^2 + \xi_+ (q^2 + p^2)] I_1 - \frac{1}{4p^2} \xi_- [k_0^4 - (q^2 - p^2)^2] I_2, \quad (19)$$

$$\mathcal{I}_{AC} = -\xi_- \frac{q_0 k_0}{p^2} I_1 - \xi_- \frac{q_0 k_0}{p^2} [k_0^2 + q^2 - p^2] I_2, \quad (20)$$

$$\mathcal{I}_B = \xi_3 I_1, \quad (21)$$

$$I_1 = \frac{q}{2p} \ln \frac{k_0^2 + (q - p)^2}{k_0^2 + (q + p)^2}, \quad (22)$$

$$I_2 = \frac{q}{2p} \left[\frac{1}{k_0^2 + (q - p)^2} - \frac{1}{k_0^2 + (q + p)^2} \right], \quad (23)$$

where $k_0 = p_0 - q_0$. Thus we need to consider the three unknown functions C , A and B with two variables p_0 and p , then the numerical analyses become much more difficult comparing to the above mentioned four dimensional cutoff case.

III. NUMERICAL SOLUTION

In this section, we numerically solve the equations with the four and three dimensional cutoff procedures by using the iteration method, then make the comparison on the obtained results.

A. Solutions with the 4D cutoff scheme

We show the numerical results of $A(P^2)$ and $B(P^2)$ in Fig. 1. One sees that $A(p^2)$ increases with respect to ξ . This can easily be understood because A has the form of $A = 1 + \mathcal{O}(\xi)$, so it becomes larger when ξ increases. On the other hand, $B(p^2)$ decreases when ξ becomes larger. This can also be understood by following discussion; B has the form of $B \propto (3 + \xi) \int dQ F(Q) \Delta(Q)$, and although $3 + \xi$ becomes larger with increasing ξ , $\Delta(Q) = 1/(A^2 Q^2 + B^2)$ decreases when A increases. Consequently, B has smaller value for larger ξ . Note that $A = 1$ always persists in the case $\xi = 0$ as obviously read from Eq. (9), which is the well-known consequence of choosing the Landau gauge. We also studied different

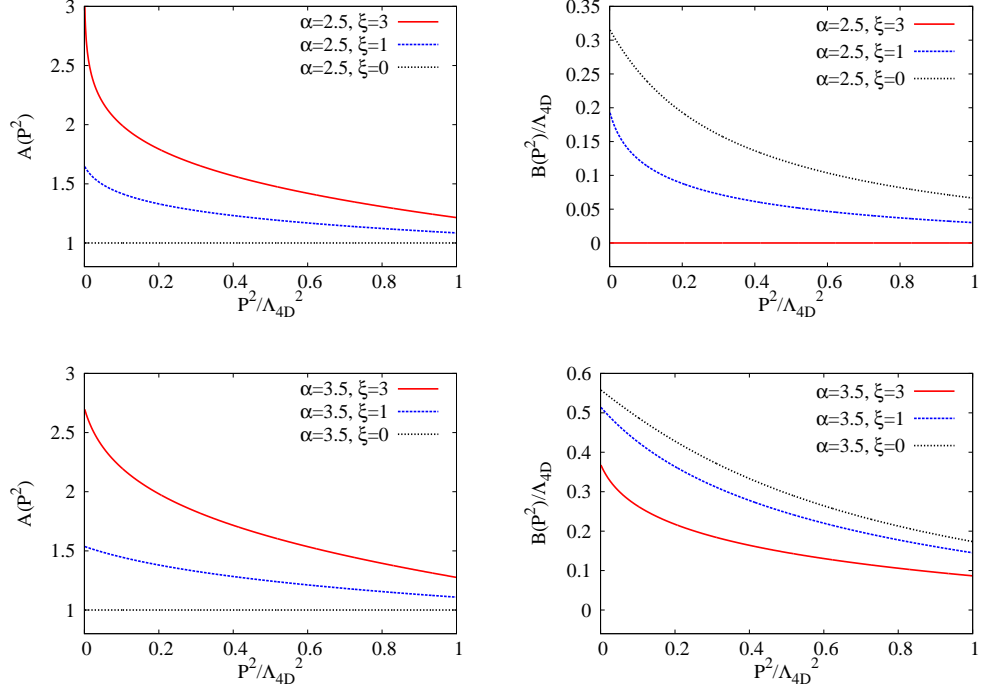


FIG. 1. Gauge dependence for $\alpha = 2.5$ and 3.5 , with $m_0 = 0$, $\delta_{4D} = 0.01\Lambda_{4D}$.

values of α , and found that the above mentioned tendency did not change, so we only showed the results with $\alpha = 2.5$ and 3.5 here.

It may also be worth studying the case with finite m_0 . Figure 2 shows the numerical results with $m_0 = 0.1\Lambda_{4D}$. One notes that the obtained values of A and B are closer for

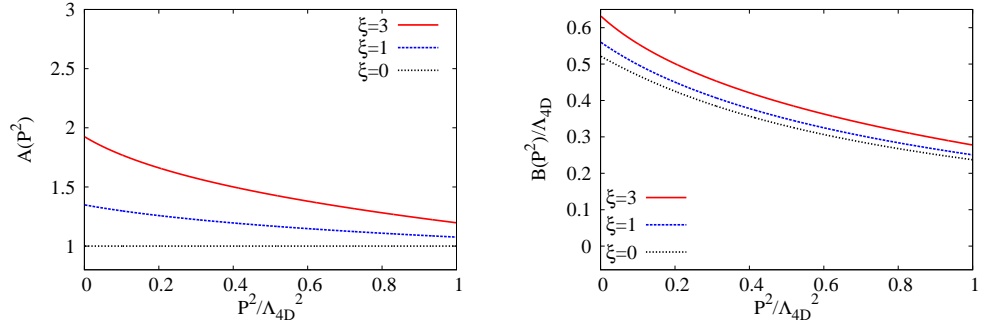


FIG. 2. Results for $\alpha = 2.5$, with $m_0 = 0.1\Lambda_{4D}$, $\delta_{4D} = 0.01\Lambda_{4D}$.

various gauge comparing to the case with $m_0 = 0$. We can read from the results that the value of B is dominated by the factor $3 + \xi$ rather than $\Delta(Q)$ because A has closer values

for this case as seen in the left panel of Fig 2.

B. Solutions with the 3D cutoff scheme

Here we show the numerical results of $C(p_0, p)$, $A(p_0, p)$ and $B(p_0, p)$ for various values of ξ with the three dimensional cutoff. Figures 3 and 4 display the solutions for $\alpha = 2.5$

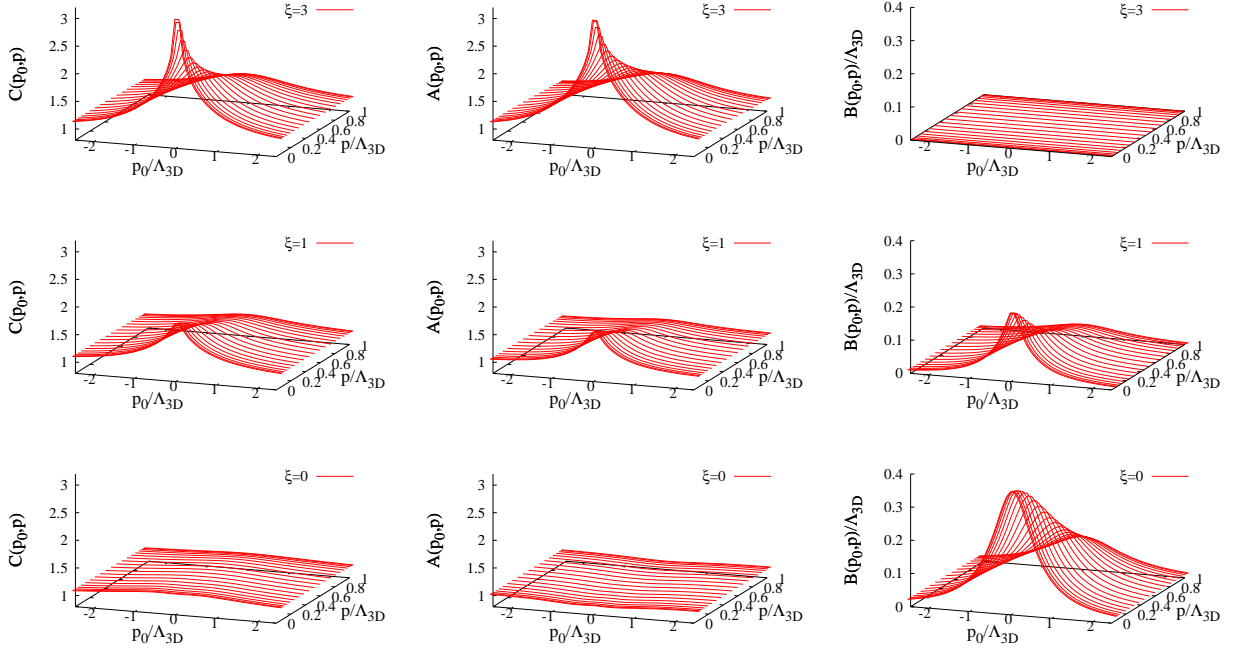


FIG. 3. Gauge dependence for $\alpha = 2.5$, with $m_0 = 0$ and $\delta_3 = 0.01\Lambda_{3D}$.

and 3.5 with the gauges, $\xi = 0, 1$ and 3. In performing the integral in q_0 direction, we set the lower and upper limit as $\int_{-\Lambda_0}^{\Lambda_0} dq_0$ with $\Lambda_0 = 10\Lambda_{3D}$ because it is technically difficult to directly take the infinite range. We have numerically confirmed that the solutions are not affected by the choice of the cutoff Λ_0 if we take large enough value for it such as $\Lambda_0 > 5\Lambda_{3D}$.

From the obtained curves, one sees the similar tendencies that the renormalization factors A and C are large when ξ is large, while the mass factor B is small for larger ξ . It may be interesting to note that A and C are close to 1, but deviate from 1 in the case of the Landau gauge, $\xi = 0$, which comes from the effect of the separation of the 4D momentum P to 3D momentum (p_0, p) .

We next check the effect of the finite bare mass $m_0 \neq 0$. Figure 5 shows the solutions with $m_0 = 0.1\Lambda_{3D}$ and $\alpha = 2.5$ for various gauges. Again, one sees the similar qualitative

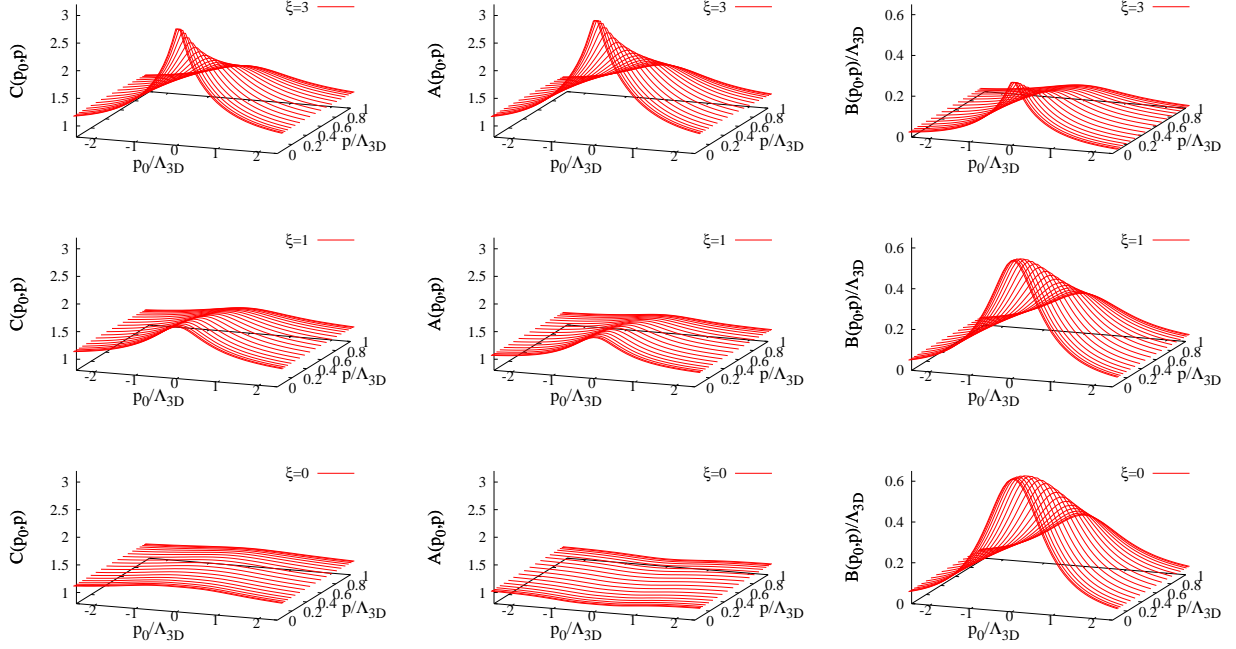


FIG. 4. Gauge dependence for $\alpha = 3.5$, with $m_0 = 0$ and $\delta_3 = 0.01\Lambda_{3D}$.

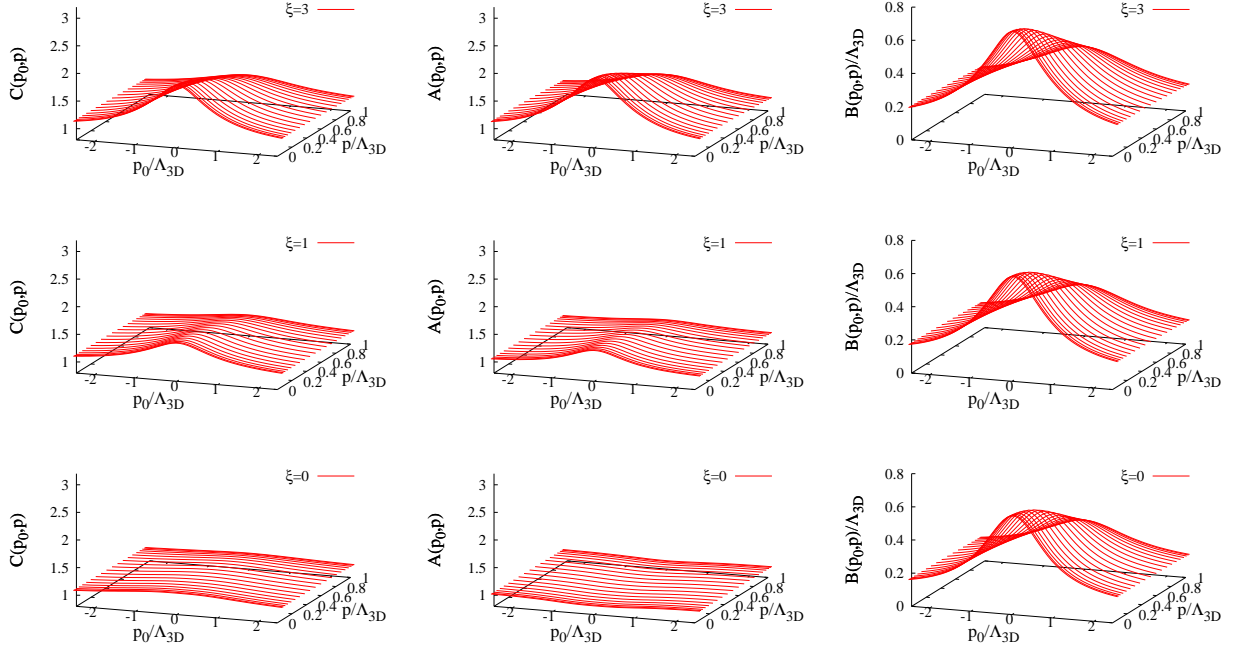


FIG. 5. Results for $\alpha = 2.5$, with $m_0 = 0.1\Lambda_{3D}$ and $\delta_3 = 0.01\Lambda_{3D}$.

tendency with the 4D case; the renormalization factors A and C become close to 1 comparing

to the results with $m_0 = 0$, and the mass factor B increases with ξ .

C. Comparison between 4D and 3D cutoff regularizations

We have seen above that the qualitative feature of the solutions with the 4D and 3D cutoff regularizations are similar. It may also be interesting to show the quantitative comparison between these regularizations, although the direct comparison is not possible since the variables are different in two methods. In Figs. 6 and 7, we align the results of $A(P)$

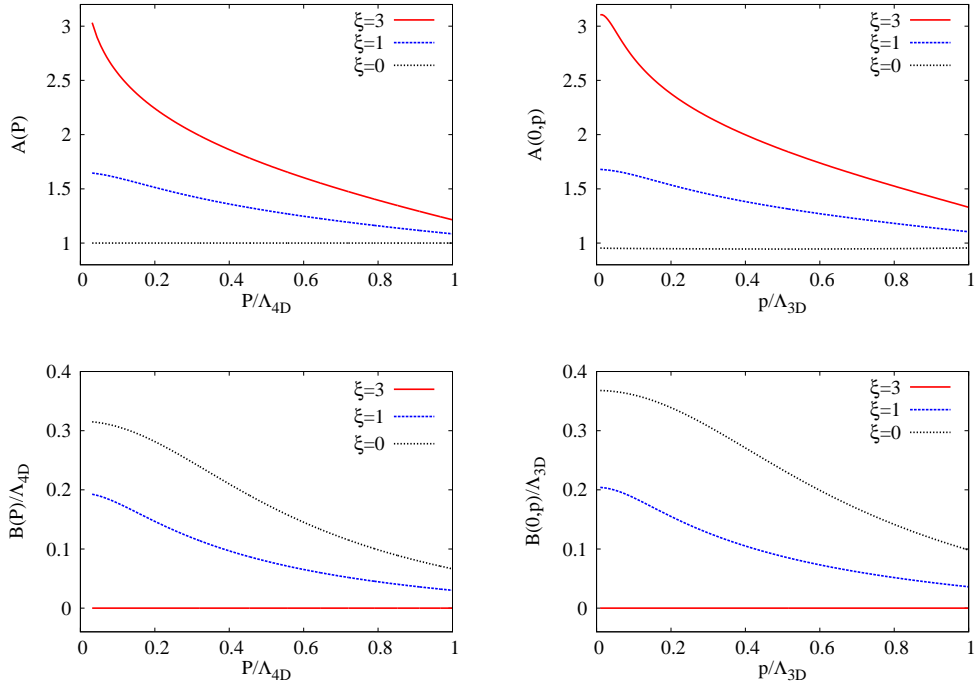


FIG. 6. $A(P)$, $B(P)$ in the 4D cut (left), and $A(0,p)$, $B(0,p)$ in the 3D cut (right) for $\alpha = 2.5$.

and $B(P)$ as the functions of P in the 4D cutoff, and $A(0,p)$ and $B(0,p)$ as the functions of p in the 3D cutoff.

We find that the results between two regularizations do not alter considerably, while the deviations between different gauges are rather large. Then we read that, concerning on the solutions on the momentum direction, the regularization dependence is not serious compare to the gauge dependence.

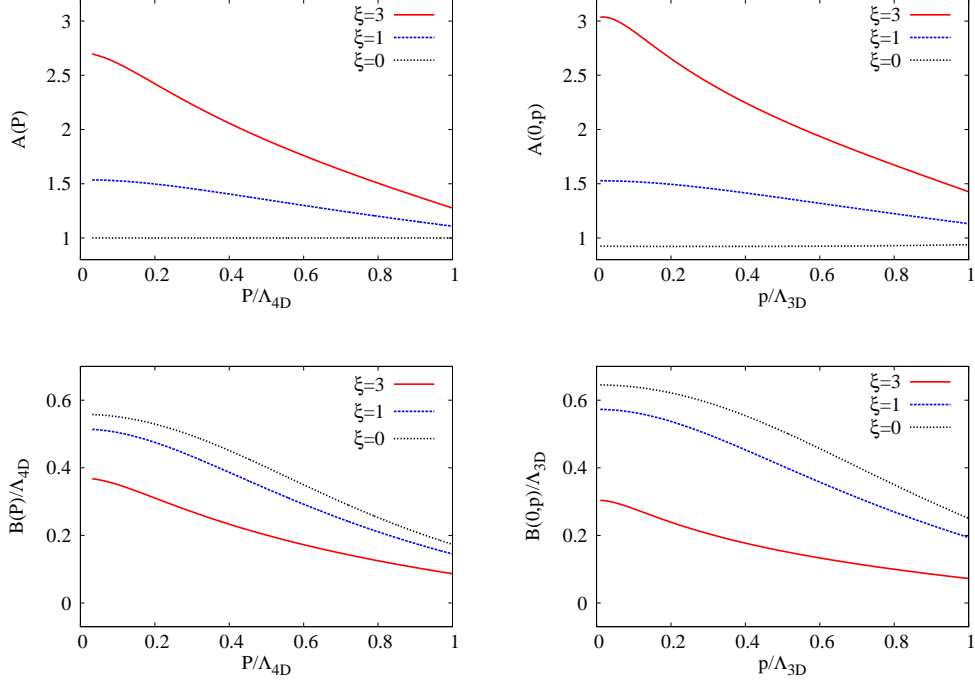


FIG. 7. $A(P)$, $B(P)$ in the 4D cut (left), and $A(0,p)$, $B(0,p)$ in the 3D cut (right) for $\alpha = 3.5$.

IV. CONCLUDING REMARKS

We have studied the regularization dependence on the quenched SDE in general gauge through applying the four and three dimensional cutoff methods in this paper. The characteristic technical difference lies on the number of variables where only one momentum, P , exists in the 4D cutoff, on the other hand there appears two different variables, p_0 and p , which makes the numerical analysis challenging. The obtained results between the 4D and 3D cutoff methods show that the regularization dependence on the solutions is not drastic comparing to the one on the gauge parameter. This indicates that both the regularization prescriptions can nicely be adopted for the analysis on the SDE.

The gauge dependence seen in the results are due to the applied approximations in deriving the equations. Especially, the tree-level approximated form of the photon propagator in Eq. (2) is crucial when we consider the gauge dependence since the gauge parameter manifestly appears in the equations. Therefore, for the sake of obtaining the gauge independent solutions as indicated by gauge theories, the general analyses, such as the ones based on the unquenched equations [5, 6], and the generalized vertices [7], are important.

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